# EXISTENCE, MEANING AND THE LAW OF EXCLUDED MIDDLE. A DIALOGICAL APPROACH TO HERMANN WEYL'S PHILOSOPHICAL CONSIDERATIONS

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## Abstract

Intuitionistic logic is often presented as a proof-based approach to logic, where truth is defined as having a proof. I shall stress another dimension which is also important: that of the constitution of meaning. This dimension of meaning does not reduce to proof, be it actual or potential, as standard presentations of intuitionistic logic put it. The law of excluded middle sits right at the junction between these two dimensions, proof and meaning: in intuitionistic logic, there is no proof for the law of excluded middle, but the law of excluded middle is a proposition that does have meaning. It is thus a problematic case. Better understanding how these two levels, meaning and proof, dissociate and interact is the purpose of this paper. I contend that the dialogical framework, a logical framework developing first and foremost intuitionistic logic (though it can also accommodate classical logic), allows to separate these two levels, meaning and proof, and show how the level of proofs rests on the level of meaning. In this respect, the law of excluded middle becomes a meaning-constitutive principle, even if it is neither proved nor refuted. The dialogical framework can thus integrate the philosophical considerations of Hermann Weyl's "intuitionistic episode" of the 1920s, which, I contend, already present a similar distinction between the level of meaning and the level of existence.

## Introduction

In his 1921 paper, "On the New Foundational Crisis of Mathematics," Hermann Weyl presents the contemporary situation in mathematics, which results from the discovery of antinomies of set theory at the turn of the 20th century, using a geopolitical metaphor: mathematics is like a realm with different provinces, some central, others remote. He describes the antinomies of set theory as "border conflicts concerning only the most remote provinces of the mathematical realm, and in no way endangering the inner

1My thanks go to the anonymous reviewer, Clément Lion, and Shahid Rahman whose remarks have helped me improve this paper. Many thanks also to Peter Gillies whose patient reading of an earlier draft helped clear the English.

soundness and security of the realm and its proper core provinces"[2]. Thus, according to Weyl, these famous antinomies were not really considered as fomenting a foundational crisis in mathematics: mathematicians dealt with them without touching the core provinces of mathematics. Weyl points out that these antinomies were "symptoms" of an evil that touched the core provinces of mathematics, and welcomes the work of L. E. J. Brouwer, a Dutch mathematician and philosopher, who tried to eradicate that evil from the whole realm of mathematics. The result was intuitionistic mathematics, a complete revision and reconstruction of mathematics. No provinces were spared. Brouwer's intuitionistic mathematics differs from the usual classical mathematics both in the set of valid theorems and in its philosophical principles. The most famous move Brouwer did was to reject the law of excluded middle, which is valid in classical mathematics (a statement is either true or false, without any third option: $A$ or not $A$). This rejection has serious consequences, since all the indirect proofs rest on the law of excluded middle and on the law of double negation elimination (not not $A$ entails $A$), also rejected in intuitionistic mathematics.

During the 1920s, Weyl was very interested in the foundational debate in mathematics and actively defended Brouwer's intuitionism. But then he moved on to other fields of inquiry, without however disowning intuitionism. Dirk van Dalen speaks in this regard of an "intuitionistic episode in Weyl's career"[3]. Because of this short active interest, and despite having been instrumental to the development of intuitionistic mathematics, it was not Weyl who went down in history for promoting intuitionism, but Arend Heyting, a student of Brouwer, who developed in the 1930s a formalization which became the standard version of intuitionistic logic (the Brouwer-Heyting-Kolmogorov interpretation).

In this paper, I will briefly present Brouwer's intuitionistic mathematics and the problem of meaning and proof that stems from the principles of intuitionism. I will then, in part one, propose an interpretation of Weyl's papers that solves this problem by dissociating the level of meaning and the level of existence. Finally, in part two, I will present the dialogical framework which clearly dissociates the level of meaning and that of existence, thus providing a solution to the problem of meaning and proof that reclaims Weyl's philosophical considerations, and producing the law of excluded middle

2 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), in P. Mancosu (Ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, 1998, pp. 86.

3 D. van Dalen, "Hermann Weyl's Intuitionistic Mathematics", *The Bulletin of Symbolic Logic, 1*(2), 1995, p. 145.

as a meaning-constitutive principle.

## §1. Brouwer's intuitionistic mathematics

Brouwer's revision of mathematics starts by clarifying what the object of mathematics is. Intuitionistic mathematics does not concern facts or the external world, but mental constructions: it deals solely with the mathematician's activity when doing mathematics. "Mathematics is, as Brouwer occasionally puts it, more of an activity than a doctrine."[4]

> "In this case [intuitionistic logic], the field of application for the rules of logic is not the world as the totality of facts, but rather the world as seen in terms of specific kinds of scientific human activities."[5]

It is the fact that intuitionistic mathematics deals only with mental constructions that guarantees the exactness of mathematics, according to Brouwer, not some logical foundation, nor the checking out of the consequences of mathematics in empirical observations.[6] Saying that intuitionistic mathematics deals with the mathematician's mental activity entails that it deals with infinity: infinity as succession in the sequence of natural numbers (adding always one more), and infinity as division in the notion of the continuum (dividing something continuous in always two more).[7] Thus, the foundation of intuitionistic mathematics lies in the idea of infinity. This is central for understanding Brouwer's rejection of the law of excluded middle: he accepts the law of excluded middle in *finite* contexts, but considers it illegitimate in *infinite* contexts. In other words, Brouwer considers that mathematicians legitimately use the law of excluded middle when they *project* a finite context upon the infinite context they are dealing with when doing mathematics, and actually deal only with these finite contexts.[8]

> "Brouwer explicitly states that only by unjustified extrapolation of logical principles from those which correctly describe the general relations among

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4 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), in P. Mancosu (Ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, 1998, p. 98.

5 K. Lorenz, *Logic, Language, and Method on Polarities in Human Experience: Philosophical Papers*, De Gruyter, 2010, p. 5.

6 L. E. J., Brouwer, "Consciousness, philosophy, and mathematics" (1948), in P. Benacerraf & H. Putnam (Eds.), *Consciousness, philosophy, and mathematics* (2nd ed.). Cambridge University Press, 1983, p. 78.

7 Ibid., p. 80.

8 L. E. J. Brouwer, "On the significance of the principle of excluded middle in mathematics, especially in function theory" (1923), in J. van Heijenoort (Ed.), *From Frege to Gödel, A Source Book in Mathematical Logic, 1879-1931* (pp. 334-342). Harvard University Press, 1967, p. 336.

> propositions on finite domains to those that allegedly regulate propositions on infinite domains, could it happen that $A \vee \neg A$ is accepted as valid."[9]

It is always possible in a finite context to determine whether or not a certain property holds of certain objects. Take for instance the decimal expansion of $\pi$ down to a certain $n$th place, as great as you wish. Once the limit is fixed and we have limited the infinite decimal expansion of $\pi$ to a finite portion, we can determine *whether or not* the sequence 0123456789 belongs to this portion of the decimal expansion of $\pi$. One only has to go through all the digits of the decimal expansion and check whether or not this sequence arises. It may take a very long time, but it is a doable (finite) task, it is possible to complete it. There is a clear *yes* or *no* answer in this finite context, and the law of excluded middle holds. This is however not the case in an infinite context: take away the limit to the decimal expansion of $\pi$, and we may never know whether or not the sequence 0123456789 belongs to it. If we find the sequence, we know it belongs to it. If we can prove that such a sequence would entail some absurd conclusion, we know it does not belong to it. But if we cannot prove it is impossible and we do not find it, we still have to keep on generating the infinite decimal expansion (it could be further on), and we will not be able to say *"no,* it is not there". There is thus no clear *yes* or *no* answer. The law of excluded middle does not hold in such a case, because in the 1920s we did not have either the answer that *yes* such a sequence belongs to the decimal expansion of $\pi$, since we had not yet found it, nor the answer that *no* it does not belong because we had checked all the instances (which is impossible since there are an infinite number of them) or because we had proved that it was impossible for this sequence to belong to it.[10]

According to Brouwer's intuitionism, propositions are a matter of *experience*: the truth of a proposition cannot be simply *stated,* it has to be *experienced*. This experience is the mental construction that mathematics is about. Thus, for Brouwer, "there are no non-experienced truths"[11].

> "To experience a truth is to experience that a certain construction has succeeded. This experience may be either direct, in case the subject has actually carried out the construction, or indirect, in case the subject sees that a certain

9 K. Lorenz, *Logic, Language, and Method…*, op. cit., p. 3.

10 We now know, since 1997, that it does. See http://www.cecm.sfu.ca/~jborwein/brouwer.html. This was an example often used by Brouwer and Heyting. See for example A. Heyting, *Intuitionism. An Introduction.* North-Holland, 1956, p. 115.

11 L. E. J., Brouwer, "Consciousness, philosophy, and mathematics" (1948), op. cit., p. 90.

> construction method will, if followed through, give the direct experience sought after."[12]

In this regard, language is simply a "nonmathematical auxiliary"[13], a tool that helps memory and allows one to communicate one's ideas to someone else. A proof is not something written down or said out loud, it is the carrying out of mental steps that brings about the expected conclusion: "a proof, then, primarily is a sequence of mental acts in which a certain experience was brought about"[14]. Thus, in the case of the law of excluded middle, the assertion $A \vee \neg A$ will be true only if it is experienced to be true, that is if one has a proof for A or a proof for $\neg A$. If one has neither, like in the case of the decimal expansion of π, then one cannot say that $A \vee \neg A$ is true, it is thus not a valid principle in intuitionistic mathematics. Interestingly enough, one cannot either say that the law of excluded middle is false, for that would mean one could prove that to have either a proof of A nor a proof of $\neg A$ is impossible (it would entail an absurdity), which is not the case either. So, the law of excluded middle is rejected as a valid principle, but it is not refuted either.

As Lorenz stresses[15], Brouwer's 1923 paper was challenging the usual value-definite way of characterizing propositions, according to which a proposition has meaning when a truth-value (true or false) can be assigned to it, determined by the truth-value of its constituent. As Brouwer stresses in his rejection of the law of excluded middle, the union of true propositions and false propositions does not cover all the propositions. There are propositions that are neither true nor false, such as those neither proven nor disproven (such as the Goldbach conjecture, the presence of the sequence 0123456789 in the decimal expansion of *π,* or future contingent propositions), and yet nobody claims that such expressions are not propositions. So, these propositions are not value-definite, they are not clearly either true or false, and yet they are still propositions. Lorenz concludes that we need a better characterization of propositions than value-definiteness, which means discarding classical logic.[16]

12M. Van Atten, *On Brouwer*, Wadsworth, 2004, p. 10. Retrieved from https://halshs.archives-ouvertes.fr/halshs-00792153

13 L. E. J., Brouwer, "Consciousness, philosophy, and mathematics" (1948), op. cit., p. 81.

14 M. Van Atten, *On Brouwer*, Wadsworth, 2004, p. 16.

15 K. Lorenz, *Logic, Language, and Method…*, op. cit..

16The difference between intuitionistic mathematics and intuitionistic logic is a matter of generality: a logical theorem is, according to Heyting, "but a mathematical theorem of extreme generality" A. Heyting, *Intuitionism. An Introduction.* op. cit., p. 6.

## §2. The problem of proof and meaning

Intuitionistic logic is a proof-based approach to logic, where truth is defined as having a proof. This conception of truth stems from the above considerations that there are no non-experienced truths, and that intuitionistic mathematics deals with mental constructions, which are either constructions of objects (for instance a certain natural number) or of proofs, which allow to have the constructed object in mind or to experience the truth of a proposition (a theorem). Thus,

> "a mathematical statement is intuitionistically true if there exists an (intuitionistic) proof of it, where the existence of a proof does not consist in its platonic existence in a realm outside space and time, but in our actual possession of it."[17]

The notion of truth in intuitionism is thus captured by the existence of a proof. A proposition is then often defined as what would count as a proof for that proposition, what would make it true (in the intuitionist sense of having a proof):

> "an understanding of a mathematical statement consists in the capacity to recognize a proof of it when presented with one; and the truth of such a statement can consist only in the existence of such a proof."[18]

In this regard, the meaning of a proposition is on the same level as the proof of that proposition. One does not necessarily need to actually possess the proof in order to understand the meaning, it is enough to know what would count as a proof. For instance, we may not be able to prove that the sequence 0123456789 belongs to the decimal expansion of $\pi$, but we may understand what it means in the sense that if we are presented with the proof that it belongs, we will recognize it is a proof of the proposition, and we will experience the truth of it. For the law of excluded middle $A \vee \neg A$, one needs to know what would count as a proof of $A$ or what would count as a proof of $\neg A$ in order to understand the meaning of $A \vee \neg A$. For the law of non-contradiction $A \wedge \neg A$, one needs to know what would count as a proof of $A$ *and* what would count as a proof of $\neg A$,

17 M. Dummett, "The philosophical basis of intuitionistic logic" (1975), in P. Benacerraf & H. Putnam (Eds.), *The philosophical basis of intuitionistic logic* (2nd ed.). Cambridge University Press, 1983, pp. 120-121.

18 M. Dummett, *Elements of Intuitionism* (2nd ed.), Oxford University Press, 2000 (1977), p. 4.

but except if one is an adept of paraconsistent logic, it will not be possible to have these two proofs together. Göran Sundholm exposes the link between meaning and truth by summarizing Dummett's argument from "Language and Truth" in *The Seas of Language*:

> "1. In order to find the condition for a proposition p to be true, a general characterization of truth is applied to p.
>
> 2. In order to apply the general characterization to p, one needs to know what proposition p is.
>
> 3. In order to know what proposition p is, one must know the condition for p to be true.
>
> 4. Thus, in order to apply the general truth-characterization to the proposition p, one must already know the outcome of this application."[19]

Thus, meaning and truth are intimately linked, to the point where Dummett expressed scepticism on the possibility of clearly defining either one:

> "What the argument shows is that the concept of truth is intricately bound up with the concept of meaning; no philosophical elucidation of either concept is to be had which does not at the same time provide an elucidation of the other one."[20]

Instead of going into the different solutions to this problem formulated by Dummett, re-formulated by Sundholm in terms of truth-makers, or the difference of Per Martin-Löf[21] between actual and potential truth, I would like to stress that this problem (and its solutions) arises when meaning and proof (or truth) are put on the same level. This is however not necessary, not even for intuitionistic logic. Another conception of meaning and proof puts them on two separate levels, where proof is built out of meaning and meaning is defined independently of its proof. Such an approach to meaning and proof is

19 G. Sundholm, "Existence, Proof and Truth-Making: A Perspective on the Intuitionistic Conception of Truth", *Topoi, 13*(2), 1994, p. 119.

20 M. Dummett, *The Seas of Language*, Oxford University Press, 1993, p. 118; quoted in Sundholm, op. cit., p. 119.

21 P. Martin-Löf, "A Path from Logic to Metaphysics", in G. Corsi, G. Sambin (Ed.), *Atti del Congresso Nuovi problemi della logica e della filosofia della scienza, Viareggio, 8-13 gennaio, 1990*, vol. 2, Logica, 1991, pp. 141-149.

provided by Lorenzen and Lorenz' dialogical logic, an interaction-based logical framework that separates the level of meaning (the play level) from the level of truth, validity, or proof (the strategy level). This dialogical framework stems from Lorenzen's operative logic[22], which delves in the foundations of mathematics and logic and has an intuitionist approach:

> "With his *Introduction to Operative Logic and Mathematics*, which first appeared in 1955, Paul Lorenzen became an exponent of an approach to the foundations of logic and mathematics, which is both formalistic and intuitionistic in spirit. *Formalistic* because its basis is the purely syntactical handling of symbols — or "figures", as Lorenzen preferred to say —, and *intuitionistic* because the insight into the validity of admissibility statements justifies the laws of logic. It is also intuitionistic with respect to its result, as Heyting's formalism of intuitionistic logic is legitimatised this way." [23]

It should be noted that Weyl himself was aware of Lorenzen's work in operative logic and welcomed it.[24]

In the dialogical framework, which will be further presented in part two, a proposition is not defined in terms of truth-conditions (be it a classical or an intuitionistic notion of truth) but it is defined in terms of dialogue-definiteness:

> "Fully spelled out it means that for an entity to be a proposition there must exist a dialogue game associated with this entity, i.e., the proposition A, such that an individual play of the game where A occupies the initial position, i.e., a dialogue D(A) about A, reaches a final position with either win or loss after a finite number of moves according to definite rules: the dialogue game is defined as a finitary open two-person zero-sum game. Thus, propositions will

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22 P. Lorenzen, *Einführung in die Operative Logik und Mathematik* (Vol. 78). Berlin, Heidelberg, Springer, 1955.

23 P. Schroeder-Heister, "Lorenzen's operative justification of intuitionistic logic", in M. van Atten et alii (Ed.), *One Hundred Years of Intuitionism (1907–2007): The Cerisy Conference*, Springer, 2008, p. 214.

24 D. van Dalen, "Hermann Weyl's Intuitionistic Mathematics", *The Bulletin of Symbolic Logic, 1*(2), 1995, p. 166.

> in general be dialogue-definite, and only in special cases be either proof-definite or refutation-definite or even both which implies their being value-definite." [25]

A proposition may thus be constructively defined without any reference to its proof. What is more, a dialogue needs to end, it is a finite process that ends with either a win or a loss. The law of excluded middle (here win or loss) is in this regard fundamental to this meaning-constitutive process that are dialogues. In this regard, I will contend that the law of excluded middle is not truth-definite, but meaning-constitutive.

The present paper intends to use the dialogical framework and its dissociation of the level of meaning and of proof as a key for understanding Weyl's 1920 papers, especially his separation of the level of meaning and the level of existence which is instrumental for his rejection of the law of excluded middle.[26] Thus the present interpretation of Weyl's philosophical investigations in the foundational debate (that I shall be presenting in a first part) is backed by the dialogical framework (which I shall present in a second part). The method carried out in this research uses the dialogical framework as a heuristic tool for the interpretation of philosophical texts.

## Part 1. Hermann Weyl: meaning and existence

In 1921, Hermann Weyl published the mathematical pamphlet "On the New Foundational Crisis of Mathematics"[27], which made much impression in the community of mathematicians and logicians.[28] In this paper and his 1925 paper, he famously advocates in favour of intuitionism by going deep into the philosophical foundations of

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25 K. Lorenz, "Basic Objectives of Dialogue Logic in Historical Perspective", *Synthese,* 127 (255), 2001, p. 258.

26My approach to Weyl's texts thus stems from logical considerations. I have endeavored to justify the proposed interpretation of his texts through his own texts, and not on external considerations. The mathematical and phenomenological aspects of Weyl's work, which I have not dealt with in this paper, should however not be overlooked. See Van Atten et al., Brouwer and Weyl, op. cit.

27 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), In P. Mancosu (Ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s* (pp. 86-118). Oxford University Press.

28 D. van Dalen, "Hermann Weyl's Intuitionistic Mathematics", op. cit.

intuitionism. Weyl searches for the point of departure between intuitionistic mathematics and classical mathematics. He finds it to be on the conception of infinity that is endorsed: intuitionistic mathematics endorse a dynamic conception of infinity, whereas classical mathematics endorse a static conception of infinity. These two divergent conceptions of infinity have bearings on how quantified statements[29] are to be understood: he goes to the very meaning of 'there is' (existential statement) and 'for all' (universal statement), which is something particular of his approach, his "second personal innovation in intuitionistic mathematics"[30]. The two different ways of understanding these quantified statements is linked to the two different conceptions of infinity, and leads on the one hand to rejecting the law of excluded middle (intuitionistic mathematics), and on the other to accepting it (classical mathematics). I will suggest that to understand Weyl's point on quantification, which is his justification for the rejection of the law of excluded middle, we must distinguish two levels, that of meaning and that of existence.

### §1. Two conceptions of infinity: dynamic and static

For Weyl, the fundamental difference between classical and intuitionistic mathematics lies in their two diverging conceptions of infinity. His 1925 paper, "The Current Epistemological Situation in Mathematics," starts by this distinction on two conceptions of infinity, distinction which operates according to him all through the history of mathematics: "from Anaxagoras to Dedekind" as the title of that section stipulates.[31] The first version of the conception of infinity is dynamic, whereas the second is static.[32]

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29 In this paper, I have not stressed the difference between proposition, judgment, and statement, used variously by the various authors quoted.

30 D. van Dalen, "Hermann Weyl's Intuitionistic Mathematics", op. cit., p. 156, pp. 158-159.

31H. Weyl, "The Current Epistemological Situation in Mathematics" (1925), in P. Mancosu (Ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, 1998, p. 123.

32See H. Weyl, "On the New Foundational Crisis of Mathematics", op. cit., pp. 94-95. Weyl gives many names to these two conceptions, but the separation seems to be the same throughout the varying nomenclature. As far as I can tell, we have these two series of names that refer respectively to what I will call the *dynamic conception of infinity* and the *static conception of infinity*, following one of Weyl's terms and what seems to be the core idea behind each conception: 1. Dynamic, Becoming, Heraclitus, continuous, Though Weyl does not mention it, these two conceptions overlap Aristotle's distinction (*Phys.* III 6, 206a14-206a18) between potential infinity (dynamic conception) and actual infinity (static conception).

1. According to the first (dynamic) conception of infinity, infinity is an ongoing process, it is the process of *becoming* as Weyl often calls it.[33] traces this conception of infinity back to Anaxagoras, in the following fragment: "Neither is there a smallest part of what is small, but there is always smaller"[34]; this fragment of Anaxagoras stresses the *process* in infinity, something that is ongoing and never completed, not a totality with everything contained in it.

2. According to the second (static) conception of infinity, infinity is a quantity of things that exist independently of our attempts to count them, which allows us to consider the totality of these existing things. Weyl refers to this conception of infinity as *being*.

From the dynamic (1) point of view, this static (2) conception is assuming that an endless process has been carried out to the end, i.e. it is assuming that something that cannot be done is actually done, thus grounding their rejection of this static (2) conception. That is the intuitionist standpoint which Weyl publicly endorses in 1921.

"A sequence is created by arbitrarily choosing the individual numbers one by one. *The result of these infinitely many acts of choice is present as finished.* Of a finished infinite se-

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process, freedom, all, instruction; 2. Static, Being, Parmenides, atomistic, totality, law, existence, abstract. These conceptions of infinity can probably be understood as conceptions of processes: 1. the dynamic conception of a process looks at the *ongoing process*, whereas 2. the static conception of a process looks at the *result of the process*. The difficulty resides in the fact that he sometimes opposes the two conceptions, and sometimes brings them together: for instance in 1921 he opposes the atomistic conception (2) and the continuous conception (1) of mathematics, saying they are incompatible (Weyl, "On the New Foundational Crisis of Mathematics", op. cit., p. 99) ; but in 1925 he does not seem to oppose these two conceptions, since he puts existential statements on the side of Being and law (2), and universal statements on the side of Becoming and freedom (1) (Weyl, "The Current Epistemological Situation in Mathematics"; op. cit., p. 134) thus suggesting that *both* conceptions, static *and* dynamic, are at work in each conception of mathematics, intuitionistic and classical. These two series should be compared to the description of the intuitionist's choice sequences in Van Atten et al. (2002; p. 10-11), which has also this ambiguity: the two conceptions are present in choice sequences between the lawlike sequence (2) and the non-lawlike sequence (1), but they are still both on the dynamic side of ongoing processes (1).

33 H. Weyl, "The Current Epistemological Situation in Mathematics" (1925), in P. Mancosu (Ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, 1998, p. 123.

34 Fr. 3, Simplicius Phys. 164, 17 quoted in P. Mancosu (Ed.), *From Brouwer to Hilbert*, op. cit., p. 142.

quence, I can, for example, ask whether the number 1 occurs in it. *Yet this standpoint is absurd and untenable, because inexhaustibility is part of the nature of the infinite.”* [35]

For Weyl, since it is impossible to end an unending process (here an infinite sequence), it is absurd to assume that such a process in finished and take the result of this finished process as a completed object. Weyl’s division between two conceptions of infinity can be generalized to two conceptions of processes (or activities): the static conception of processes (2) does not consider the process as something ongoing, in its development. It considers the process in its result, without the limitations internal to the ongoing process, which may even have vanished when the process has completed. The dynamic conception of processes (1) insists on those limitations, and considers one cannot do away with them: a process is completed only once it has taken the internal limitations into account and has been completely carried out with them. Take for instance the proof that *there are irrational numbers a and b such that $a^b$ is a rational number.* There is a classical proof which proceeds by cases[36]. Either $\sqrt{2}^{\sqrt{2}}$ is rational, or it is not.

- If $\sqrt{2}^{\sqrt{2}}$ is rational, then we can answer the question in the affirmative by setting $a=b=\sqrt{2}$.

- If $\sqrt{2}^{\sqrt{2}}$ is not rational, let $a=\sqrt{2}^{\sqrt{2}}$ and $b=\sqrt{2}$ ; then $a^b=(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}=\sqrt{2}^2=2$ , which is rational.

Since the two cases cover all the possibilities according to the law of excluded middle (which is classically valid), and since both cases produce the conclusion that *yes, there are irrational numbers a and b such that $a^b$ is a rational number,* it is legitimate to answer "yes" absolutely. Or so goes the classical reasoning.

There is however no intuitionistic proof, cause the cases do not allow us to know, in the end, what number *a* is since we cannot decide which case is in fact the case (whether $\sqrt{2}^{\sqrt{2}}$ is a rational number or not). Saying that such a number exists but without being able to say which it is, as in the classical proof, would be, from the intuitionistic

35 H. Weyl, “On the New Foundational Crisis of Mathematics” (1921), op. cit., p.94.

36 M. Van Atten, *On Brouwer*, op. cit., p. 2.

perspective, doing as if something ongoing was fixed and finished.

Since for intuitionists “mathematics is the science of the infinite”[37], these two conceptions naturally yield two different ways of doing mathematics. Thus, dealing with mathematics supposes one to be clear with one’s own conception of infinity: intuitionistic mathematics only accept infinity as an unending process (dynamic conception, 1) and reject the static conception (2), whereas classical mathematics also accept a static version of infinity (2). Weyl contends (part IV, reaffirmed in 1932) that only Brouwer’s intuitive mathematics can deal with the genuine continuum, thus obtaining real numbers (say $\sqrt{2}$) without having to assume an infinite process to have been completed: they are an approximation that is always more precise than what is required at any moment, ¨*a number that is given only approximatively, yet one for which the degree of approximation can be pushed beyond any limit”* [38]. Thus a real number is not an object entirely determined and contained in itself (like a natural number, say 2), it is a process whose determination depends on a previously fixed precision requirement; real numbers in intuitionistic mathematics are more an activity of the mathematician than a real independent object.

In 1921 Weyl’s enthusiasm for intuitionistic mathematics combined with the “pamphlet” nature of the paper seems to point to a resolution of the tension between the two conceptions of infinity in favour of the dynamic conception.

> “It would have been wonderful had the old dispute led to the conclusion that the atomistic [static, 2] conception as well as the continuous [dynamic, 1] one can be carried through. Instead the latter triumphs for good over the former. It is Brouwer to whom we owe the new solution of the continuum problem.”[39]

However, Weyl was a physicist as well as a mathematician, and was keen on finding a way to reconcile those two conceptions of infinity; or, if that proved to be impossible, to

37 H. Weyl, “The Current Epistemological Situation in Mathematics” (1925), op. cit., p. 123.

38 H. Weyl, “On the New Foundational Crisis of Mathematics” (1921), op. cit., p. 93.

39 *Ibid*., p. 99.

give each conception its appropriate domain of application. In 1932, he explicitly opts for two different ways of doing mathematics: when doing pure mathematics, one should opt for Brouwer's intuitionist way, since the dynamic conception (1) is most adequate for dealing with human activities (mathematics considered as a mental activity); but when doing mathematics applied to the natural sciences, one should opt for Hilbert's classical axiomatization, since the static conception (2) is more adequate for dealing with the physical world.[40]

> "My opinion may be summed up as follows: if mathematics is taken by itself, one should restrict oneself with Brouwer to the intuitively cognizable truths and consider the infinite only as an open field of possibilities; nothing compels us to go farther. But in the natural sciences we are in contact with a sphere which is impervious to intuitive evidence; here cognition necessarily becomes symbolical construction. [...] from this higher viewpoint which makes the whole of science appear as one unit, I consider Hilbert to be right."[41]

Weyl thus considers that natural sciences require some ontological assumptions that allow for instance to consider the world as given and complete in itself, for which classical mathematics (and logic) would be better suited.[42] In this regard, accepting or rejecting the law of excluded middle is less a matter of choosing a set of principles that should be valid (accepting the law of non-contradiction and rejecting the law of excluded middle, for instance), than a matter of adopting a certain philosophical perspective on

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40 Especially if one considers the objects of study in the physical world as a given totality, or as finite sets of totalities: then, we are in the domain of the finite, not the infinite or continuum; thus the two conceptions, dynamic (1) and static (2), are not to be opposed anymore. Contrary to an infinite process, a *finite* process can be carried out to the end.

41 H. Weyl, "The Open World: Three Lectures on the Metaphysical Implications of Science" (1932), in P. Pesic (Ed.), *Mind and Nature. Selected Writings on Philosophy, Mathematics, and Physics* (2nd ed.), Princeton University Press, 2009, p.80.

42 This view is shared by Lorenz, who stresses "the difference in points of view between the proponents of classical logic and the proponents of effective [intuitionistic] logic" (K. Lorenz, *Logic, Language, and Method on Polarities in Human Experience: Philosophical Papers*, De Gruyter, 2010, p. 5). Classical logic is appropriate for dealing with a totality of facts, whereas intuitionistic logic is appropriate for dealing with a system of human activities.

what it is that logic is doing, if it is capturing an activity in its process or rather something that is assumed already to be there, somewhere, somehow. It is choosing between the static and the dynamic conception of processes.

But even in the 1920s, when promoting the dynamic conception (1), Weyl makes some concessions to the static conception (2). The continuum and infinity *qua* infinity are adequately rendered through the dynamic conception (1); but as a physicist, Weyl seems compelled to recognize an ordinary realist tendency to consider things as being in themselves, such as the "real external world, which is believed to exist in itself and to possess a composition determined in itself"[43]. In this realist, static conception (2), if 'there is' something, then it must be there *somewhere*, even if we cannot reach it, build it, or understand it. In this regard, real numbers are considered as real, as *there* somewhere, somehow, and not simply as an approximation always more precise than what our current need may be (such as real numbers according to the dynamic take on infinity).

Such an ordinary realist tendency is what makes us consider the law of excluded middle as a valid principle, for we usually believe that "[this] state of affairs does or does not obtain; the judgment is in itself true or not true – without change or wavering and without any possibility of a point of view mediating between the two opposed positions".[44] It is probably because of this realist tendency of ours that Weyl tries to fit together the two conceptions, static and dynamic, rather than oppose them completely. He finds a balance between the two at the level of what it means to say 'there is' (existential statement) and 'for all' or 'every' (universal statement): he calls the dynamic side (1) *Becoming* and *freedom*, and the static side (2) *Being* and *law*.

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43 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), op. cit., p. 89.

44 *Ibid.* What Weyl does with this realist tendency of ours is very close to the explanation of Brouwer of our illegitimate tendency in favor of the law of excluded middle: applying the law of excluded middle is legitimate in finite domains, and applying it in infinite domains is legitimate only because we are then projecting finite domains on infinite domains, we proceed *as if* we were dealing with "(*possibly partly unknown*) finite discrete systems that for specific known parts are bound by specific laws of temporal concatenation" L. E. J. Brouwer, "On the significance of the principle of excluded middle in mathematics", op. cit., p. 336.

> "The 'there is' ties us to [the realm of] *Being* and *law*; the 'all' places us in [that of] *Becoming* and *freedom.*" [45]

An existential statement (*there is*) places us on the static side (2), which considers things that exist independently of our attempts to know them, things that exist completely determined and contained in themselves. To take one of Weyl's images[46], this is like saying 'there is a treasure': we may not know where it is, how to get to it or who put it there, but either it is there wherever it is, or it is not at all and there is no treasure. Our use of existential statements tends to follow this pattern of considering the domain of application as a completed whole determined in itself. On the other hand, universal statements (*all* or *every*) place us on the dynamic side (1), which only considers some activity as an ongoing process; when the domain of application is finite, it is possible to run through each case one by one, as a police officer through his files (another of Weyl's images)[47]; but when the domain is infinite, we only have the characteristic properties of the elements (like being a number, being even, and so on), we do not have them all in front of us, like all the files of the officer. For instance, when speaking of even numbers, we only have the characteristic properties of the elements, that they are numbers, that they are even; we cannot have *all* the even numbers before us (2, 4, 6, … ): we can give a finite list, but then will have to use '…' to mark the never-ending process of generating more even numbers.

In the 1920s, Weyl explicitly endorses Brouwer's intuitionism, which brings him to separate classical mathematics from intuitionistic mathematics on the basis of their different conception of infinity (respectively a static one (2) and a dynamic one (1)). In adopting intuitionistic mathematics, Weyl has to reject the law of excluded middle, but he has his own justification for rejecting this law, it is not the same as Brouwer's or the later Intuitionists: for Weyl, assertion and negation in existential and universal

45 H. Weyl, "The Current Epistemological Situation in Mathematics" (1925), op. cit., p. 134.
46 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), op. cit., p. 97.
47 *Ibid*., p. 87.

statements do not form complete disjunctions[48], which entails a rejection of the law of excluded middle. This stems from Weyl's particular conception of existential and universal statements, which I believe overlaps Aristotle's distinction between meaning and existence (*Posterior Analytics,* B 1-2, 89b23-90a6), and to which we shall now turn.

## §2. Two levels: meaning and existence

In 1921 Weyl converts to Brouwer's intuitionistic mathematics. He presents his own previous theory of analysis whose foundations lie in the static conception of infinity (2), a theory he had already formulated in 1918 in *Das Kontinuum*; but he does not stand by that theory anymore, and presents it only to provide contrast with Brouwer's dynamic approach of infinity (1). Looking back at his 1918 attempt, Weyl publicly forsakes the static approach (2) in favour of the dynamic approach (1):

> "For this order is in itself untenable, as I have now convinced myself, *and Brouwer – that is the revolution*! The reason I have presented the basic idea of my theory here is that it and Brouwer's theory bring out most sharply the old contrast between the atomistic [2] and the continuous [1] conceptions, and that the contrast between the two shows forcefully where the problem lies, and what needs to be done."[49]

But these diverging conceptions do not simply entail different ways of doing mathematics, for the two would be "equally possible"[50]; they also entail different conceptions of what it is to say "there is" (existential statement) and "every" (universal statement). Thus, for Weyl, it is in this last difference that the main divergence between classical and intuitionistic mathematics lies.[51] Weyl's great contribution to intuitionism was to insist on the meaning of judgments in general, and in particular on the meaning

48 *Ibid.*, p. 97.

49 Ibid., pp. 98-99.

50 Ibid., p. 98.

51There are recurring discussions on the meaning of existence in intuitionistic and constructive logic; see for instance Sundholm, *Constructive Recursive Functions, Church's Thesis, and Brouwer's Theory of the Creating Subject*, op. cit.

of universal and existential statements[52].

Even though Weyl does not explicitly refer to Aristotle, the core of his reflexion seems to rest on the Aristotelian division in two kinds of all that we seek (or understand): 1. whether a thing is (*ei estin*), and 2. what that thing is (*ti estin*) (Aristotle, *Posterior Analytics* B1-2, 89b23-90a6, translation and commentary by Barnes). I will now show that this division is relevant for reading Weyl. The problem here is not historical, but conceptual. The point is not to know whether or not Weyl was indeed directly influenced by the Aristotelian division when he undertook his reflexions leading to his 1921 paper.[53] The point is to know whether or not this division is operative in his work. If my reading of Weyl is correct, the Aristotelian division operates at the level of the distinction between extensional and intensional definiteness.[54]

### §2.1 Extensional and intensional definiteness

Extensional definiteness concerns the existing instances, those actually produced which constitute a complete totality in itself, as if laid out in front of us (like the police officer's files). It is for instance the individual numbers 2, 4, 6. Intensional definiteness on the other hand concerns clear and unambiguous meaning, that is, the characteristic properties of all the possible instances, e.g. 'being an even number'. The first are concerned with existence, the second with meaning. The first suppose an answer to the kind of question *whether it is,* the second suppose an answer to the kind of question *what it is*. Here is the main passage supporting my interpretation of Weyl as implementing the Aristotelian distinction between existence and meaning. Let us

52 D. van Dalen, "Hermann Weyl's Intuitionistic Mathematics", *The Bulletin of Symbolic Logic, 1*(2), 1995, p. 156.

53Though it must be said that Weyl often refers to Ancient philosophies and to Aristotle, as far as I know he does not explicitly refer to this well-known passage of the *Posterior analytics*. However, his two conceptions of infinity are not explicitly linked to Aristotle's potential and actual infinity either, even though they overlap.

54 When speaking of definite concepts, Weyl separates two cases of definiteness: when we consider the existing objects that fall under the concept and when we consider the characteristic properties of the objects falling under that concept. I will follow Weyl in saying that the first concept is extensionally definite, and I will say that the other concept is intensionally definite, though Weyl does not use this expression.

therefore dwell a bit on it.

> “[A] It may well always be that the sense of a clearly and unambiguously determined object concept [*Gegenstandsbegriff*] assigns to the objects of the nature expressed by the concept their sphere of existence. [B] But this does not make the concept an *extensionally definite* one; that is, it does not ensure that it makes sense to consider the *existing* objects that fall under the concept as an ideally closed, in itself determined and delimited totality. [C] This cannot be so if only because the wholly new idea of existence, of being-there [*Dasein*], is added here, while the concept itself is only about a nature, a being-such-and-such [*So-sein*].”[55]

In [A] and [B] the intensional definiteness of a concept [A] is opposed to its extensional definiteness [B]. In [C] the opposition is clearly marked as existence vs. nature, which I understand as existence vs. meaning: these terms seem to be answers to Aristotle’s questions: is it? (answer: its *Dasein*, its being-there) and what is it? (answer: its *So-sein*, its being-such-and-such).[56]

Weyl points out in [A] that a well-defined concept defines the *sphere* of existence within which any object of such a nature would have to be *if* they *were* to exist, since otherwise they would simply not be objects of that nature. A nature is what is expressed by a concept, it is *being-such-and-such,* what a concept in itself (and not through its instances) is about. A nature can thus be understood as the meaning of the concept. A well-defined concept is determined clearly and unambiguously through its properties (intensional definiteness), and so it determines the sphere of existence of all the possible objects that might fall under this concept. But [B] this intensionally definite concept defines only the *sphere* of existence, through the *properties* of the object, it does not define the *actual existence* of the objects of that nature. This would be the case for an extensionally definite concept: the actual existence of the objects would be given with

55 H. Weyl, “On the New Foundational Crisis of Mathematics” (1921), op. cit., p. 89; my A-B-C division, the German words are the original terms inserted by the translator; italics are the author’s.

56 Weyl’s rendering of the Aristotelian distinction would thus be on the side of answering, whereas Aristotle was distinguishing ways of searching, of questioning.

their concept, the existing objects would be available, they could be considered as a given totality that is both determined and delimited in itself. This is the idea of existence, of *being-there*.

For instance, the concept of natural numbers can be defined by recursion in the following way.

$$\begin{cases} 0 \textit{ is a natural number} \\ \textit{if } n \textit{ is a natural number, its successor } s(n) \textit{ is a natural number} \end{cases}$$

This concept is clearly and unambiguously determined through these two clauses. But in itself, this definition only determines a nature (the meaning), it does not provide the actual instances of natural numbers such as 2, 3, 4, which are the result of the mathematician's activity of generating instances by applying the definition. Generating the instances out of the definition produces "the wholly new idea of existence, of being-there" [C]. When one carries out the instructions provided in the definition, one builds these numbers in one's mind, one actually has these objects in mind. One has to build these numbers through an activity: 0, 1, 2, 3, etc. are instances of natural numbers that exist because they have been constructed in one's mind. This concept however is not extensionally definite, for all the natural numbers do not form an "ideally closed, in itself determined and delimited totality" [B]: the process of producing the instances is endless, and we shall never have all the actual instances as if in front of our eyes. This definition of natural numbers determines in itself the *sphere* of existence of all natural numbers, but not their actual existence; something more than the definition needs to produce their actual existence. This something more is the mathematician's activity that carries out the definition. Thus, meaning and existence are two separate things, and existence adds something that meaning in itself does not have: actual existence.[57]

The difficulty at this point is to understand how this distinction of existence and

57 Would that entail that meaning is potential existence? Such a conclusion would go against the separation of meaning and existence that I am arguing for. Actual and potential existence are a matter of existence, not of meaning. Here "actual" is taken as a way to stress existence, not as a qualifier of existence that is opposed to some potential existence.

meaning combines with Weyl's take on existential and universal statements. Indeed, in what immediately follows the passage [C], Weyl explains why these two levels (what I take to be meaning and existence) tend to be confused by calling upon our above-mentioned realist tendency – which, as I understand, projects a finite structure on the world, considered as a totality existing in itself and possessing "a composition determined in itself"[58]. In such a domain, the law of excluded middle holds, and a question about meaning as well as about existence both have a clear *yes* or *no* answer.

> "If the concept *B* is, in particular, extensionally definite, then not only will the question "Does *x* possess the property *E*?" have a clear and unambiguous sense for an arbitrary object *x* falling under *B*, but also the existential question "Is there an object falling under *B* that possesses the property *E*?"[59]

Consider for instance the natural numbers up to a certain *n*, as great as you wish. This concept is extensionally definite, and the question "does *x* possess the property of being even?" can be answered by *yes* or *no* for any natural number *x* inferior to *n*. The same can be done for the existential question "Is there a natural number inferior to *n* that possesses the property of being even?" since it is possible to go through all the instances one by one and determine in the end whether or not there is such an instance (like the police officer through his files). In other words, in such finite contexts, when the concepts can be extensionally definite, the law of excluded middle holds.

But one must conclude that in infinite contexts, concepts cannot become extensionally definite through an enumeration of the instances: the enumeration process would never end, we would never have the completed totality before us. Concepts then can only be intensionally definite, and the law of excluded middle does not hold in the dynamic perspective (which, contrary to the static one, does not project a finite realist structure on the infinite domain at stake).

To put the pieces together, the static conception of infinity (2), which produces classical mathematics, considers any domain as if it existed in itself and could be dealt

58 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), op. cit., p. 89.

59 *Ibid*.

with as a completed totality. The law of excluded middle then holds, and intensional and extensional definiteness go together. The dynamic conception of infinity (1), however, which produces intuitionistic mathematics, considers infinite domains to be different from finite domains. In infinite domains, intensional and extensional definiteness need to be distinguished; this produces a hiatus between *yes* and *no* answers to existential questions (is there...?) or universal questions (do all...?): *yes* or *no* do not produce a complete disjunction anymore, and thus the law of excluded middle cannot hold.

### §2.2 Existential and universal statements

In *finite* domains, the static conception (2) can be produced, once the process has been carried out to the end (which is possible since the domain is finite). But in *infinite* domains, concepts cannot be extensionally definite, and the question "does *x* possess the property *E*?" can only be answered once *x* has been determined. If the instance *x* has not been determined, we are dealing with the universal question "do every object falling under *B* have the property *E*?" which is a matter of meaning (*yes* answer) or existence (*no* answer):

- to answer "*yes*, every object falling under *B* has the property *E*," one must show that this property *E* follows necessarily from the fact that the object is in the sphere of existence defined by the nature *B* of the object (see [A] above), and we are here making a universal statement;

- whereas to answer "*no,* every object falling under *B* does not have the property *E,"* one must either provide an instance proving that there is an object that falls under *B* but that does not have the property *E*, and we are here making an existential statement, or prove that having the property *E* is in contradiction with falling under *B*, and we are making a universal statement (*no object falling under B has the property E)*.

In such a case (universal question), the law of excluded middle does not hold: all the instances could have a property *E* without it being necessary, i.e. without having this property because of the definition of the nature of the object. For instance, all the natural numbers could have the property "being pretty according to my nephew". They could *all* have this property, but the property is not necessarily derived from the definition of a natural number.

On the other hand, the existential question "*is there* an object falling under *B* that possesses the property *E*?" can be answered affirmatively only once an instance has been discovered (level of existence, producing an existential statement), but otherwise not, since all the instances cannot be checked one by one. Answering *no* would mean either that we have carried the infinite process to the end, which is impossible, or that we have shown that having this property is incompatible with the definition of the nature of the object (level of meaning, producing a universal statement): having the property would require for the object not to be in the sphere of existence determined by its definition, which is an absurdity (see [A] above). Thus, answering *yes* or *no* to the existential question is respectively 1. providing an instance thus proving that it is so (existential statement), or 2. proving that such an instance entails an absurdity (universal statement).

For the existential question (*is there?*) just like for the universal question (*do all?)*, we are not dealing with a complete disjunction between the *yes* and *no* answers. Indeed, for the existential question, it could be that there are no such instances with this property without this property being incompatible with the definition. This is for instance the case with the sequence 0123456789 in the decimal expansion of π: in the 1920s, we did not know if this sequence was in the decimal expansion of π, but it was not incompatible with the definition of π that it be there. It is only later on, when this sequence had actually been found in the decimal expansion of π, that one could say "*yes*, it is there" (existential statement). So the question "is it there, yes or no?" cannot be transformed into the complete disjunction "*yes* it is there, or *no* it is not, and there is no other option" (law of excluded middle), since saying "*yes*, it is there" is an existential statement (level of existence) that requires having found an actual instance; whereas saying "*no*, it is not" is a universal statement (level of meaning) that requires showing that it being there would be in contradiction with some property of the definition of π. Thus, the *yes* and *no* answers to the existential (*is there?*) and universal (*do all?*) questions do not form complete disjunctions. We are here in the traditional intuitionist rejection of the law of excluded middle. Weyl insists however on the meaning of existential and universal statements, and not on the temporal aspect of something not yet determined.[60]

> "The expression 'there is' commits us to Being and law, while 'every' releases us into Becoming and freedom. Given that the range of cases in which the one or the other claim holds (i.e., there is a sequence of property *E*, and every sequence has property *E*, respectively) is not determined in itself, indeed, given that the concept of a sequence has to be interpreted completely differently in the two cases, it would be absurd to think of a complete disjunction in this context. In this way one will understand Brouwer's claim that there are no grounds for believing in the *logical principle of the excluded middle*. Although I would prefer to say that it is no longer possible to say of the two statements that one is the negation of the other." [61]

Thus, the existential statement "there is an object falling under the concept *B* with the property *E*" does not have the same interpretation as the universal statement "there is no object falling under the concept *B* with the property *E*". For *there is*, one must provide an instance (existence level), but for *there is no*, one must show that an absurdity would follow from inserting the property in the sphere of existence of the objects (meaning level). So, with "there is", the law of excluded middle does not hold, we cannot say "there is or there is not an object falling…". The same goes with "every object falling…", which requires a determination at the level of meaning, and "not every object…", which requires a determination at the level of existence. In this regard, Weyl provides a justification for rejecting the law of excluded middle by going into the very meaning of universal and existential statements, which I have argued requires the separation of the level of existence and the level of meaning. And this separation finds its full import in the dynamic conception of infinity, not in the static one (where intensional and extensional definiteness do not need to be distinguished).

60 D. van Dalen, "Hermann Weyl's Intuitionistic Mathematics", *op. cit.*, pp. 158-159

61 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), op. cit., p. p. 96, my underlining.

If we take a step back and look at Weyl's global reasoning, we can say the following. Intuitionists reject the law of excluded middle; they are right to do so because intuitionism is based on a dynamic conception of infinity, and this dynamic conception requires different procedures for universal and existential statements, procedures which use extensional definiteness for existential statements (*there is* and *not all*) and intensional definiteness for universal statements (*all* and *there is no*), and thus do not form a complete disjunction anymore, rendering the law of excluded middle meaningless.

### §2.3 Proper judgments, judgment instructions, judgment abstracts

Now, the important step in Weyl's reasoning, which I believe links in this scheme the difference between meaning and existence, is to say that existential statements are meaningless if they are not linked to proper judgments or to universal judgments.

> "Indeed, in the context of number sequences and the laws that determine them *in infinitum*, we have already said: If we have succeeded in constructing a law with property *E*, then we are justified in making the claim that there are laws of kind *E*. Only the *successful* construction can provide the justification for this; the mere *possibility* is out of the question. Yet what sort of judgment, which by itself is meaningless, indeed which only acquires a meaning on the basis of the successful proof guaranteeing its truth, is this? The point is that this is no judgment at all, but a judgment abstract. With this, I believe, its character is clearly described and the proper sense of the concept of existence elucidated."[62]

Existential statements, as such and on their own, do not have any meaning, they are like saying "there is a treasure" without providing the map or saying where it is.[63] The

62 H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), op. cit., p. p. 98.

63 According to the dynamic conception, which yields a notion of existence different from the static conception. For more on the difference between classical and intuitionist existence, see for instance G . Sundholm, "Constructive Recursive Functions, Church's Thesis, and Brouwer's Theory of the Creating Subject: Afterthoughts on a Parisian Joint Session", In J. Dubuc & M. Bourdeau (Ed.), *Constructivity and Computability in Historical and Philosophical Perspective* (Vol. 34), Dordrecht, Springer, 2014, pp. 8-12.

meaning is provided through proper judgments, such as *2 is an even number* (everything in the judgment is determined), which are finite, and through universal statements, such as *every natural number dividable by 2 is even,* which can range over infinite domains and provide instructions for building proper judgments. Meaning requires universal statements, but does not provide the existing instances; existence is something added to meaning. One must then distinguish proper judgments, which concern given (or constructed) instances, such a 2, and existential statements (judgment abstracts), which mentions neither the instance in question nor the way to produce such an instance. Thus, *2 is an even number* is not the same as saying *there is an even number*. The first is a proper judgment, the second is a judgment abstract, which, according to Weyl, has no meaning on its own, and is barely a judgment at all. Thus, existential statements need to be built out of proper judgments and universal statements (judgment instructions) in order to have meaning. Bare existence, as simply *there exists*, is an abstraction and puts us on the static conception's side of our ordinary realist tendency; bare meaning, as general properties, is an instruction, and puts us on the dynamic conception's side.

In this respect, the law of excluded middle seems to be for Weyl meaning-constitutive: it is a requirement for clear and unambiguous meaning to know whether or not a property can be said of a nature or not. This however is not a description of some state of affairs: saying that a property can be said of a nature (meaning level) is not saying that it is true to say it is so, for that would be introducing the "wholly new idea of existence" (or being) in the picture. So either 1. we are at the level of Weyl's proper judgments and the law of excluded middle *for these judgments* holds because we are in a *finite* and fully determined context, or 2. we are at the level of meaning without any consideration of truth or existence and the law of excluded middle in its general form holds as a requirement for clear and unambiguous meaning, or 3. we are at the level of existence and here the law of excluded middle does not hold because universal and existential statements do not form a complete disjunction, "it is no longer possible to say of the two statements that one is the negation of the other" (see above). In its generalized form, the law of excluded middle would thus be a meaning-constitutive principle, but not an adequate description of what is (existence or truth) from the perspective of the dynamic conception of infinity.

This separation of the level of meaning and the level of existence has some importance in intuitionistic logic, to which we shall now turn. This separation does not have meaning for classical logic which determines the meaning of a proposition through its truth conditions, thus conflating the two levels. However, this kind of separation is not usual in intuitionism, which tends, as we have seen, to look first at proofs (the truth of a proposition being defined as actually having a proof for it) and determines meaning as what would count as a proof if we had one. I contend that this is not exactly the distinction that Weyl implements: in speaking of a judgment abstract that is meaningless if not backed with a successful proof, I believe he is not dividing proofs in actual and potential proofs, where truth would be defined as actually having a proof, whereas meaning would be defined as potentially having a proof. I rather believe Weyl is separating existence from meaning, saying that existential statements *need* meaning, they do not have meaning on their own. Thus, existential statements need to rest on proper judgments and judgment instructions (universal statements).

## Part 2. The dialogical framework: meaning and proof

We started with the problem of meaning and proof in intuitionistic logic: if meaning is defined in terms of proof, we risk falling into a circle. The solution adopted here is to separate the level of meaning and the level of proof. We have seen that, according to the proposed interpretation, Weyl separates the level of meaning and the level of existence. For Weyl, bare existential statements (such a *there is an even number*) do not have meaning on their own, they are judgment abstracts, and they need to rest on proper judgments (such a *4 is an even number*) or judgment instructions that need to be carried out in order to have proper judgments (such as *every successor of a natural number is a natural number,* which can be carried out into *1 is a natural number* since *0 is a natural number* and *1 is the successor of 0*). The link between existence and proof is however not clearly stressed. The dialogical framework explicitly differentiates the level of meaning and the level of proof, and allows to integrate Weyl's distinction between the level of meaning and the level of existence (a proposition is true or proved when there exists a winning strategy for it in the dialogical framework).

The dialogical framework is a logical framework based on different dynamic interaction processes between two players, the proponent and the opponent: 1. the meaning of the logical constants (conjunction, disjunction, negation, and so on) is determined through rules of interaction between two players; 2. the meaning of a proposition is determined by dialogue-definiteness, that is, the capacity for the two players to carry out a play according to the interaction rules determining the meaning of the logical constants at stake as well as the structural rules for plays; 3. the proof for a proposition is built out of plays by looking at the proponent's play options for each of the opponent's possible choices during a play. The whole structure of this dialogical framework is thus centred on a dynamic process provided by interaction.

I believe for this reason that it is a perfect framework for rendering Weyl's basic concepts of intuitionistic mathematics, which we have seen to be steeped in the dynamic conception of infinity. What is more, the dialogical framework is not a particular logic, but a framework that can support different logics, such as intuitionistic logic or classical logic. The difference between the two can be formulated within this dynamic framework, and I would like to show how close the resulting account is to Weyl's.

Let us first present in an informal way the importance of separating the level of meaning and the level of proof, and how it is done in the dialogical framework. This will allow a presentation of the law of excluded middle as an operative principle that should only be *carried out* and not *talked about,* accounting for its prevalence in what counts as a meaningful statement without nonetheless being proved. We shall then present the dialogical framework which backs these considerations: first the play level (§2) and the difference between classical and intuitionistic logic in the framework (§3), then the strategy level (§4), before finishing on remarks on meaning and existence in the dialogical framework (§5).

## §1. The law of excluded middle is meaning-constitutive but not truth-definite

The dialogical framework allows for a formalization of mainstream intuitionism together with Weyl's distinction between meaning and existence. Weyl goes down to the very meaning of existential and universal statements, and, according to my interpretation of his texts, he separates existence from meaning and puts the law of excluded middle on the side of meaning and not of existence; it is meaning-constitutive but not truth-definite: if something is well defined, then its meaning is clearly and unambiguously determined, it is easy to answer *yes* or *no* to the question *is it so-and-so?* (it is then intensionally definite). But this question is not a matter of existence but of the nature of the thing. The question about existence is a question of *being-there*, and this is not in the definition of the concept or property. Thus, a logical framework having these two separate dimensions, existence and meaning, would need to have two different procedures, one for determining the conditions under which a judgment is meaningful, another for determining the conditions under which it is true.

These two separate procedures are not usually provided in the logical frameworks. It is for instance not the case when the meaning of the logical constants is provided by truth tables (classical logic), nor when the truth is equated to the holding of the proof and the meaning of a proposition is determined as what would count as a proof (usual presentation of intuitionism). In both cases meaning and existence are basically on the same level, and existence is no longer a "wholly new idea"[64].

In the dialogical framework, the meaning of a proposition is determined at the level of plays, i.e. at the level of an actual exchange of arguments between two parties. One is said to know the meaning of a proposition when one can carry out such an exchange of arguments to the end, regardless of who wins the argument. Thus, knowing the meaning of a contradiction amounts to knowing how to challenge and defend a conjunction ('and') and a negation, and to be capable of arguing to the end. Knowing the meaning of

64 One could of course argue that this would be Weyl's meaning of a "sphere of existence" : not the actual existence but what the defined existing thing would be if it actually were. This would be a way to show that Weyl's version of intuitionism can fit in the mainstream version of intuitionism (developed after Weyl's 1921 paper). But it would not take into account this distinction between meaning and existence that seems to be at the heart of his rejection of the law of excluded middle in 1921.

the law of excluded middle amounts to knowing how to challenge and defend a disjunction ('or') and a negation, and to be capable of arguing to the end (see below for a more precise presentation of the dialogical framework). This level is analogous to Weyl's judgment instructions which become proper judgments when the instruction has been carried out. Thus, an initial proposition has meaning, it is analogous to a judgment instruction, because one can carry out a finite exchange of arguments with someone else, starting with this initial proposition, and the argument will end with one of the two parties winning and the other losing. The choices made in the exchange of arguments are determined, and at the end of the argument, there is nothing left to further determine, we are at a stage analogous to Weyl's proper judgments. Another exchange of arguments can however take place, and the "judgment instruction" of the initial thesis can be carried out in some other determined way. What is important here, is that the meaning level is based on a *finite* process that carries out the meaning of the initial proposition, and it wins with either win or loss for each party. One does not have to win in order to know the meaning of a proposition. This is the dialogue-definiteness conception of propositions of Lorenz quoted above in the introduction, and further exposed below in §4.

The level of proof in the dialogical framework does not concern particular exchanges or arguments about one initial proposition, they concern *all* the different particular exchanges that can take place. There is a proof for the initial proposition when the proponent (the party first stating the initial proposition) has a *winning strategy*, i.e. when that party is capable of winning any exchange of arguments, whatever the other party chooses to do during the argument. Thus, a contradiction will never be won by the proponent, there is no winning strategy for it: there is no proof for it, it is not true.[65] For the law of excluded middle, there is no winning strategy for the proponent when he plays with the intuitionistic rules, but there is a winning strategy when he plays with the

65There is even a winning strategy *against* it, where it is the opponent (the other party) who wins each time, whatever the proponent may choose to do. A contradiction is thus refuted. In a play against a proposition, it is the opponent who must play according to the copy-cat rule. See below the presentation of the dialogical framework.

classical rules. In the dialogical framework, the level of proof is built on the level of meaning: the proof (the winning strategy) comes from considering all the finite exchanges of arguments and checking that none is won by the opponent. We thus enter a possibly infinite context, in which there are procedures that allow to reduce this possibly infinite amount of plays to a finite number of relevant ones. If the result of the procedure says that the opponent cannot win, then the conclusion is that *there is* a winning strategy, *there is* a proof. This is the level of existence. Now if we separate the existential statement *there is a winning strategy (proof)* from the procedure that enabled us to make that statement, i.e. if we only have the *result* of the procedure, we have a bare existential statement which has no meaning, it is a judgment abstract: *there is* a *winning strategy for a proposition* informs us as little as saying simply *there is a treasure* . The meaning is provided by all the different exchanges of arguments (the plays).

This separation of the level of meaning (the play level) and the level of proof (the strategy level) allows to have meaningful propositions even if one sometimes wins and sometimes loses the argument or always loses. Win or loss is not the same as true and false, nor as proved and refuted. They are two different levels: win or loss is a local matter, it is the ending of a particular (finite) exchange of arguments (a dialogue play); true or false, proved, not-proved, refuted are a matter of strategies, a bird's-eye view on all the possible exchanges of argument in order to conclude that *there is* a proof and *it is* true, or *it is* refuted and *it is* false, or *it is* not proved nor refuted so one can say neither that it is true nor that it is false.

In this fashion, in the dialogical framework for intuitionistic logic, the law of excluded middle is dialogue-definite (see below for the play) but not truth-definite: one can carry out a play (exchange of arguments) to the end, but the proponent does not have a winning strategy (he actually loses the play). However, the opponent does not have a winning strategy *against* the law of excluded middle, and so the proposition is neither proved nor refuted, neither true nor false. The law of excluded middle can thus be accepted as a meaning-constitutive law: it is the principle that is fundamental for a proposition to have meaning, the plays must end with either win or loss, there is no further option. This law is then a law that is *carried out* and not *talked about*.

Now, if the law of excluded middle is meaning-constitutive, and if meaning and proof (or truth) are on two separate levels with the level of proof being built out of the level of meaning, then it seems natural that the law of excluded middle cannot be proved. Indeed, if a proof rests on the meaning of what is at stake, and that meaning is constituted by the law of excluded middle (win or loss), then talking about the law of excluded middle and trying to prove it supposes that very thing that is being talked about. As an applied meaning-constitutive principle is thus beyond the reach of proof. With these notions in mind, let us now present the dialogical framework in more details.

## §2. Introducing to the dialogical framework

The dialogical framework is a dynamic framework through and through: it is based on dialogues, which supposes interaction, be it with someone else or with oneself (in which case one is arguing with oneself, taking up the two different roles, challenging an idea and then defending it). This dynamic aspect makes the dialogical framework naturally suited for intuitionistic logic and its procedural conception of proof. It also provides the means to differentiate, within a logical framework, Weyl's two levels of meaning and of existence. The drawback however of the dynamic nature of dialogical logic is that the framework is difficult to present in written, static form: it requires active reading to recreate in one's mind the dynamic interactions that are taking place. These interactions are of two kinds: challenges and defences. Once the proponent $P$ has stated the thesis (the proposition that is the subject-matter of the dialogue), each move in the dialogue is either a challenge of one of the other player's previous statements, or a defence against a previous challenge. The moves need to be read in the order they are made, starting with the proponent. In order to help the reader, a description of the moves is written on the outer side of the dialogue tables. Here is a play for the law of excluded middle (with intuitionistic rules). Let us present it step-by-step, presenting the dialogical rules on the way. Let us then contrast this play with the play on contradiction ($A$ and not-$A$), on the law of non-contradiction (not-($A$ and not-$A$)), and finally contrast the two plays for the law of excluded middle, one with the intuitionistic logic rules and one with the classical logic rules. This should be enough to give the reader a good feeling for what the dialogical framework is about. The main idea to keep in mind is that we are here talking about a game between two players, and we actually carry that game out in the plays.

The two players (or roles) are the proponent *P* and the opponent *O*. The game starts when the proponent *P* states a thesis, the first statement of a play; each player in turn then has a choice of available challenges and of defences. The first player who can neither challenge nor defend at his turn loses the play. The basic idea behind this framework is to test how far one can stand behind a statement (the thesis) when facing a critical opponent. If one can withstand any challenge or force the other to make a statement that he cannot defend, then that statement is logically true (it is proven, the proponent has a winning strategy).[66]

In order to test the law of excluded middle ($A \vee \neg A$, LEM) in the dialogical framework, we build a table where the two players, the proponent *P* and the opponent *O,* face each other (two columns, each for recording a player's moves). The proponent *P* states the thesis. It is move 0, written in the outer column.

**LEM play, move 0:**

| *O* | | *P* | | |
|---|---|---|---|---|
| | | $A \vee \neg A$ | 0 | *P stating the LEM* |

Once *P* has stated the thesis (here the LEM), it is *O*'s turn to play. By convention, *O* is she and *P* is he. *O* has at this point only one option, she must challenge the thesis. Challenges and defences always concern the main logical constants (conjunction $\wedge$, disjunction $\vee$, negation $\neg$, etc.). Here, it is a disjunction ($A \vee \neg A$).

The meaning of the logical constants is determined by the way they are challenged and defended. It is a rule-based approach to meaning in the "meaning as use" tradition.

66I will present the dialogical framework in this informal manner. For a more formal presentation, see S. Rahman et al., *Immanent Reasoning or Equality in Action. A Plaidoyer for the Play Level*. Springe, 2018, chapter 3 as a basic tutorial, chapter 4-5 as a more technical presentation of basic dialogical framework. For a more powerful version of the dialogical framework, see chapters 6-7 which present Immanent reasoning. I will only stay at the level of basic dialogues in this paper.

In this fashion, the meaning of the logical constants can be compared with the movements of chess-pieces, each piece being defined by its legal moves on the board. Weyl already used that comparison with chess-pieces[67], but to present Hilbert's purely formal logic, in which the chain of formulae has as little meaning as a game of chess. However, the dialogical framework is a branch of game theory, where games are the means for providing meaning: in games, rules and action are intimately linked, and that interrelation is the source of meaning. According to the dialogical framework, the meaning of statements requires another person capable of challenging one's grounds for stating a proposition. Meaning and justification are thus intertwined in this game of giving and asking for reasons[68] through rules defining both meaning and ways of acting.

> "To assert a proposition makes sense only, if there is someone on the other side, albeit fictitiously, who either denies or at least doubts the asserted proposition. But it is not enough merely to argue about propositions, there must exist precise stipulations on the rules of argumentation, rules which, in a way, define the exact meaning of the proposition in question."[69]

What then are the rules for the dialogical game of logic? The rules analogous to the movements of the chess-pieces are the rules defining the challenges and defences for the logical constants. For the law of excluded middle, we need the rules for disjunction and negation. Stating a disjunction commits the player to one of the two disjuncts (*X* or *Y*); which one of the disjuncts to state is the defending player's choice. The challenging player simply requests that the other player defends the disjunction by stating one of the two disjuncts.

| | Statement | Challenge | Defence |
|---|---|---|---|
| Disjunction | $X \vee Y$ | $? \vee$ | *X (defender's choice)*<br>*Y* |
| | *A player states "X or Y".* | *The other player challenges the disjunction by requesting that he states either X or Y.* | *The first player defends the disjunction by stating either X or Y, as he wishes.* |

Disjunctions can be contrasted with conjunction: when stating a conjunction (*X* and *Y*), the player commits to the two conjuncts, and so the choice as to which one he is

67 H. Weyl, "The Current Epistemological Situation in Mathematics" (1925), op. cit.

68 This expression is taken from Brandom. See for instance R.B. Brandom, *Making it Explicit: Reasoning, Representing and Discursive Commitment*. Harvard University Press, 1994, p. 167.

69 K. Lorenz, *Logic, Language, and Method on Polarities in Human Experience: Philosophical Papers*, De Gruyter, 2010.

required to state in the play is given to the challenging player.[70]

| | Statement | Challenge | Defence |
|---|---|---|---|
| Conjunction | $X \wedge Y$ | $? \wedge 1$ (challenger's choice)<br>$? \wedge 2$ | *X (respectively)*<br>*Y* |
| | *A player states "X and Y"* | *The other player challenges the conjunction by requesting X or by requesting Y, as he wishes.* | *The first player defends by stating the conjunct that was requested.* |

As for negation (not *X*, or *X* yields an absurdity), challenging a negation requires that the challenger states the proposition without the negation, and the defence is to give up the play. Thus, being forced to defend a negation amounts to losing the play.

| | Statement | Challenge | Defence |
|---|---|---|---|
| Negation | $\neg$ X | *X* | $\perp$ *("I give up")* |
| | *A player states "not X"* | *The other player challenges the negation by stating X* | *The first player cannot defend the negation, or gives up the play.* |

We can now go back to our law of excluded middle game. It was *O*'s turn to play, and she must challenge the disjunction. Each new challenge goes on a new line, with the corresponding defence inserted on the same line. *O* challenges the disjunction, it is move 1 (written in the outer column, but on *O*'s side this time).

### LEM play, move 1:

| | *O* | | *P* | | |
|---|---|---|---|---|---|
| | | | $A \vee \neg A$ | 0 | *P states the LEM* |
| *O challenges the disjunction (0)* | 1 | $? \vee$ | | | |

It is now *P*'s turn to play. Since *O* has not stated anything during the game, *P* cannot challenge any of *O*'s statements. He therefore must defend the challenged thesis (here the LEM). He must choose one of the two disjuncts, *A* or ¬*A*, and state one of them.

A final set of rules must be introduced before going any further: the structural rules. In chess, there are the rules for moving the pieces, but also the rules determining how to start a

70 Weyl's considerations of freedom and law or constraint can well fit in such a dialogical setting, freedom being what one can choose, constraint what the other chooses to do.

game, end it, who wins, and any other special rules (castling, for instance). In the dialogical framework, we have informally seen how to start, end, and play. But there is also a special rule, which is absolutely crucial: the copy-cat rule. The copy-cat rule (also known as the formal rule or Socratic rule) is a restriction put only on the proponent *P* and on elementary statements (i.e. statements without any logical constant). According to the copy-cat rule, *P* may state an elementary proposition only if *O* has already stated that elementary proposition. This restriction is what ensures formality in the dialogues. The rationale behind this rule is that challenging a complex statement is asking for the player's reasons for stating it, and the defence provides such a reason by further stating propositions that one was committing to when stating the complex proposition. An elementary statement, however, cannot be challenged (at least not in this basic version of dialogical logic), and so no reason can be brought forth backing this statement. Since a dialogue is the process of examining the thesis, when defending the thesis it is *P* who needs to be justifying all his choices,[71] and the copy-cat rule is a way of ensuring that *P* has good reasons for stating elementary propositions during the play: those statements are justified because the opponent has said so herself. These reasons are formal in the sense that it is copying the implicit reasons of *O* for stating her elementary propositions, whatever those reasons may be. It is a justification internal to each play and built out of the very interaction process of challenging and defending the statements uttered during the play.

Going back to our play on the law of excluded middle, we can see that according to the copy-cat rule, *P* is not allowed to state *A* since *O* has not stated it yet. So *P* must state the other disjunct, i.e. $\neg A$, which is complex (it is a negation). Since it is a defence, it is inscribed on the same line as its challenge. It is move 2.

### LEM play, move 2:

| *O* | | *P* | | |
|---|---|---|---|---|
| | | $A \vee \neg A$ | 0 | |
| 1 | ? $\vee$ | $\neg A$ | 2 | *P defends the disjunction by choosing one disjunct* |
| | | | | |

71If the examination process is trying to reject the thesis, we will be looking at plays *against* the thesis, in which case it is *O* who will be subjected to the copy-cat rule. Refuting a proposition is building a winning strategy (see below) for *O* out of plays against a thesis. See K. Lorenz, "Basic Objectives of Dialogue Logic in Historical Perspective", *Synthese, 127*(255), 2001, p. 257-258 and S. Rahman et al., *Immanent Reasoning or Equality in Action. A Plaidoyer for the Play Level*, Springer, 2018, pp. 180-181.

It is now $O$'s turn to play. She has no other option than to challenge $P$'s last move (move 2). It is a negation, so challenging it requires that she state the negated proposition. A challenge is written on a new line. It is move 3.

### LEM play, move 3:

| | O | | P | |
|---|---|---|---|---|
| | | | $A \vee \neg A$ | 0 |
| | 1 | ? $\vee$ | $\neg A$ | 2 |
| *O challenges the negation (2) by stating what is negated* | 3 | $A$ | | |
| | | | | |

This is the end of the play: it is $P$'s turn to play but he cannot play any further. Defending his negation requires giving up, so that he would lose the play. But he has nothing else to defend and $O$ has stated only an elementary statement which cannot be challenged. Therefore, $P$ loses.

### LEM play, whole:

| | O | | P | | |
|---|---|---|---|---|---|
| | | | $A \vee \neg A$ | 0 | *P states the LEM* |
| *O challenges the disjunction* | 1 | ? $\vee$ | $\neg A$ | 2 | *P states one disjunct (his choice)* |
| *O challenges the negation (2)* | 3 | $A$ | | | *P has no available move (intuitionistic rule)* |
| | | *wins* | *loses (intuitionistic rule)* | | |

In this play, the proponent loses when stating the law of excluded middle. The rules that are being used are basically the rules for intuitionistic logic. See below how these rules can be adapted in order to obtain classical logic (where the law of excluded middle is valid, and in dialogs where the proponent can win any play).

Compare the game for the law of excluded middle with the one for a contradiction, $A \wedge \neg A$ . Since with contradiction we are dealing with a conjunction (and not a disjunction), it is the challenger who chooses which conjunct should be stated. So, $O$ can immediately ask for the elementary statement *A*, the first conjunct; *P* then loses for he has no move left (because of the copy-cat rule forbidding him to defend the conjunction by providing the required elementary conjunct *A*).

**Contradiction play, whole:**

| | *O* | | *P* | | |
|---|---|---|---|---|---|
| | | | $A \wedge \neg A$ | 0 | *P stating the thesis* |
| *O challenging the conjunction* | 1 | $? \wedge$ | | | |
| | | *wins* | *loses* | | *P loses because he cannot say A (copy-cat rule)* |

Now for the law of non-contradiction (LNC), ¬(*A* ∧ ¬*A*), *O* has first to challenge a negation, and so to state the proposition negated, which is a contradiction. *P* cannot defend the negation, otherwise he loses the play, but *O* has stated a complex proposition which *P* can now challenge (on a new line, move 2); since it is a conjunction, *P* requests one of the two conjuncts, as he wishes, say the second. *O* then has to defend the conjunction and provide the requested conjunct, i.e. ¬*A* (move 3).

**LNC play, moves 0-3:**

| | *O* | | *P* | | |
|---|---|---|---|---|---|
| | | | $\neg(A \wedge \neg A)$ | 0 | *thesis* |
| *O challenging the negation (0)* | 1 | $A \wedge \neg A$ | | | |
| *O defending the conjunction by stating the requested conjunct* | 3 | $\neg A$ | $? \wedge 2$ | 2 | *P challenging the conjunction (1)* |
| | | | | | |

It is now *P*'s turn. He cannot challenge the negation ¬*A*, since that would require him to state *A*, which is elementary and has not yet been stated by *O*, the copy-cat rule thus prohibiting this move. It is therefore time for us to introduce the last structural rules for basic dialogues. These rules will allow us to compare intuitionistic logic and classical logic. Both concern the challenges and defences the two players are allowed to do.

The first of these additional structural rules introduces repetition ranks: *O*'s repetition rank is 1, she may challenge each statement at most once, and defend against each challenge at most once; *P*'s repetition rank is 2, he may challenge each statement at most twice, and defend against each challenge at most twice. This rule renders finite plays.

The second additional structural rule differentiates intuitionistic and classical logic. For intuitionistic logic, a player may defend only the last challenged statement (this is called “last duty first”). For classical logic, the restriction is removed.

So, for our play about the law of non-contradiction, it is *P*´s turn, and according to his repetition rank, he may challenge once more *O*’s conjunction by requesting the first conjunct, *A*. It is a challenge, so it is inscribed on a new line. It is move 4.

**LNC play, move 4:**

| | *O* | | *P* | | |
|---|---|---|---|---|---|
| | | | $\neg(A \wedge \neg A)$ | 0 | |
| | 1 | $A \wedge \neg A$ | | | |
| *O defending the conjunction by stating the requested conjunct* | 3 | $\neg A$ | $? \wedge 2$ | 2 | *P challenging the conjunction (1)* |
| | | | $? \wedge 1$ | 4 | *P challenging conjunction (1) with repetition rank 2* |
| | | | | | |

*O* has to defend this new challenge: she has to state *A* (move 5). *P* is then entitled to challenge the negation (¬*A*) by stating *A*, which *O* has just stated (so the copy-cat rule no longer restricts him from making this elementary statement). *O* has no further move and loses.

**LNC play, whole:**

| | *O* | | *P* | | |
|---|---|---|---|---|---|
| | | | $\neg(A \wedge \neg A)$ | 0 | |
| | 1 | $A \wedge \neg A$ | | | |
| | 3 | $\neg A$ | $? \wedge 2$ | 2 | |
| *O defending the conjunction by stating the requested conjunct* | 5 | *A* | $? \wedge 1$ | 4 | *P challenging the conjunction (1) with rank 2* |
| | | | *A* | 6 | *P challenging the negation (3) by stating what is negated* |
| *O has no available move* | | *loses* | *wins* | | |

This play about the law of non-contradiction does not require the classical structural rule. It is won by *P* with the intuitionistic rule. Recall the play for the law of excluded middle; it followed the intuitionistic rule.

### LEM play, whole (intuitionistic logic):

| | *O* | | *P* | | |
|---|---|---|---|---|---|
| | | | $A \vee \neg A$ | 0 | *P states the LEM* |
| *O challenges the disjunction* | 1 | $? \vee$ | $\neg A$ | 2 | *P states one disjunct (his choice)* |
| *O challenges the negation (2)* | 3 | *A* | | | *P has no available move (intuitionistic rule)* |
| | | *wins* | *loses (intuitionistic rule)* | | |

The law of excluded middle is lost by *P* with the intuitionistic rule: the last challenge is on a negation, so responding to that last challenge would be giving up the play, and *P* has no challenge to play. He has no further move and thus loses with the intuitionistic rule. The play is however won with the classical structural rule: since the players are not required to defend the last challenge, *P* can defend once more against the challenge move 1, and provide the second disjunct this time. Since *P* is using his repetition rank of 2 in order to defend a second time against this challenge, we put the challenge on the new line in square brackets, thus showing that it is not another move made by *O*. *P* is allowed to state this elementary *A* this time because *O* has just said it herself (move 3). *O* has no further move.

### LEM play, whole (classical logic):

| | *O* | | *P* | | |
|---|---|---|---|---|---|
| | | | $A \vee \neg A$ | 0 | *P states the LEM* |
| *O challenges the disjunction* | 1 | $? \vee$ | $\neg A$ | 2 | *P defends the disjunction* |
| *O challenges the negation (2)* | 3 | *A* | | | |
| *[repetition of O's challenge, move 1]* | [1] | $[? \vee]$ | *A* | 4 | *P defends the disjunction (rank 2): states the other disjunct* |
| *O has no available move left* | | *loses* | *wins (classical rule)* | | |

In this fashion, the game about the law of non-contradiction is won by *P* with intuitionistic rules, and the game about the law of excluded middle is won by *P* with classical but not intuitionistic rules.

## §3. Comparing classical logic and intuitionistic logic in the dialogical framework

The dialogical framework is built on the dynamic conception of processes: its whole structure is dynamic, from the meaning of the logical constants defined through rules of interaction to the carrying out of the plays which records the process of interaction between the two players. The copy-cat rule ensures the formality of the plays by inserting an internal limitation to the ongoing interaction process. This limitation however is not visible if one looks only at the result of the play (who won, who lost), and not at the development of the play. It is the development of the play that tells us *why* a player played that move, and *how* the proponent $P$ can defend the thesis, if at all. Now, if we follow Weyl and consider that the dynamic take on processes is at the core of the intuitionist approach, it is the intuitionistic structural rule that renders the fundamentally dynamic structure of the dialogical framework: the classical structural rule is an added generalization in order to introduce the static conception of processes in this framework. In this respect, the classical structural rule is not native to the framework but clearly imported; it adds something that was not present in the framework itself, a static aspect.

If we look at the two different plays for the law of excluded middle, the classical play does not fully respect the ongoing interaction process of justification: $P$ is entitled to defend the disjunction $A \vee \neg A$ by stating the elementary proposition $A$ only because $O$ stated $A$ when challenging $P$'s choice of $\neg A$ to defend the disjunction. What $P$ does is not against the rules, for he abides by the copy-cat rule and the repetition ranks, but he uses the process of justification of one of the disjuncts in order to justify the other disjunct. Thus, if we look at the ongoing process, when $P$ defends his disjunction a second time (move 4), he is doing as if one process of justification (that for $\neg A$) was completed in order to use it for the other process of justification (that for $A$). He thus considers that when $O$ challenged the negation ($\neg A$), she was committing herself to $A$ absolutely, not in the context of $P$'s committing to $\neg A$. He is thus forcing more committments on $O$ than the intuitionist account would accept, for such an account would insist that the two disjuncts should be *independently justified* (which is what the "last duty first" rule amounts to, among other things). It is only the attention given to the ongoing process (how the play is carried out) that can reveal in what way the two cases ($A$ and $\neg A$) are not separated, and no direct proof of each is provided.

Note that things stand differently for the law of non-contradiction: *P* is challenging *O*'s conjunction twice in a row (because of his repetition rank of 2), requesting both conjuncts in a case where she did commit to both (this is the dialogical meaning of a conjunction).

This is not to say that the classical rule should be absolutely rejected with the law of excluded middle; the two different logics may have different purposes. As we have seen, in 1932 Weyl took some distances with Brouwer's intuitionistic mathematics, without however disowning it. He simply stresses the different needs of mathematics in itself and physics, and therefore the different conceptions of justification and of infinity that are required. In this regard, accepting or rejecting the law of excluded middle is less a matter of choosing a set of principles that should be valid (accepting the law of non-contradiction and rejecting the law of excluded middle, for instance), than a matter of adopting a certain philosophical perspective on what it is that logic is doing, if it is capturing an activity in its process or rather something that is assumed already to be there, somewhere, somehow. It is choosing between the static and the dynamic conception of processes.

### §4. Existence of a winning strategy out of meaning-providing plays

Up to now, we have only considered the level of plays. The level of plays is the carrying out of individual plays about a thesis, according to the rules for the logical constants (particle rules) and the structural rules. A proposition is dialogue-definite when a play about it can be carried through to the end; this feature of dialogue-definiteness is what ensures that the proposition has meaning.

> "A proposition shall be called "dialogue-definite" under the condition that the possible dialogues on this proposition are finished after finitely many steps according to some previously stipulated and effectively applicable rules of argumentation, such that, at the end, it can be decided who has won and who has lost. Hence, dialogue-definiteness of propositions means that the relevant concept of a dialogue is decidable. And it is this concept of dialogue-definiteness that is to replace the age-old value-definiteness as the characterizing feature for linguistic expressions to be propositions."[72]

Dialogue-definiteness thus replaces the classical value-definiteness, but it is also

72 K. Lorenz, *Logic, Language, and Method on Polarities in Human Experience: Philosophical Papers*, De Gruyter, 2010, p. 11.

different from the usual intuitionist take of providing what the proof for that proposition is, or what would count as a proof for it. A dialogical play is not a proof. For a proposition to be a (meaningful) proposition, it is enough to be able to carry out a play according to the rules and determine who wins and who loses. In order to prove a proposition in the dialogical framework, we need to take all the possible plays into account. This brings us to the level of strategies.

Strategies are defined for a player, usually for *P. P* has a winning strategy when he has a way of winning, whatever *O*'s moves may be. This is analogous to the "check-mate in *x* moves" in chess: whatever the opponent's moves may be, there is a way for the player to move that will allow him to win in maximum *x* moves. Having a winning strategy for *P* is the dialogical equivalent for building a demonstration of the proposition or proving it is true.[73] The dialogical framework thus recovers the notion of proof and of validity (logical truth) at the strategy level.

To build a strategy for *P*, we must first carry out a play. If *O* wins the play, it is basically sufficient to say that *P* does not have a winning strategy for that proposition. But if *P* wins the play, then we go bottom-up in the order of moves and stop when *O* made a choice in her moves, that is, if she defended a disjunction or challenged a conjunction. At that point, we start another play, exactly the same down to the move at stake, and there *O* chooses the other option. We then carry on the play to the end. If *P* wins, then again, going bottom-up in the order of moves, we look for a choice made by *O* (and not already dealt with in another play). Each time, we consider the play in which that choice would have been different. Once all the choices made by *O* have been considered in separate plays, and if *P* was able to win each play, then *P* has a winning strategy for that proposition. Having a winning strategy thus requires going through all the different choice options for *O* and checking that *P* can win each time. The level of strategies considers all the relevant plays (those in which *O* makes a choice), whereas the level of plays considers only the ongoing plays, one by one, as they are effectively carried out.

A proposition is defined through the level of plays, as being dialogue-definite: it can have meaning without having a demonstration, i.e. without *P* having a winning strategy for it (nor *O* having a winning strategy *against* it). Meaning is defined by knowing how to play a dialogue about this proposition, and making actual choices when playing;

73 See for instance K. Lorenz, "Dialogic Logic", in W. Marciszewski (Ed.), *Dictionary of Logic as Applied in the Study of Language.Concepts/Methods/Theories* (Vol. 9),). Dordrecht, Springer, 2013, p. 118; and N. Clerbout, "First-order dialogical games and tableaux", *Journal of Philosophical Logic, 4*, 2014, pp. 785-801.

demonstration and validity (logical truth) are defined through a winning strategy for *P*, taking into account *all* the relevant plays.

Winning strategies (proof or logical truth) are built out of the play level, through a definite procedure taking all the plays into account. The level of strategies is generated out of the level of plays (meaning) through a procedure that adds something completely different from the plays, which is the consideration of all the relevant plays, that is, all the possible choices for *O* during the plays. This means that strategies as such are never effectively carried out, they are a theoretical construction allowing one to grasp why the proposition is true, how it is built, and why one can assert it with the full backing of a proof. But any examination of that proposition would have to be effectively carried out at the play level. Even starting another play and playing differently is not being at the level of strategies but staying at the level of plays. Only plays are effectively carried out. That *P* wins or loses a play is not saying that the proposition is true or false. Contingent truth is not positively dealt with in the dialogical framework. Meaning and proof (or truth) are two separate things with two different procedures in the dialogical framework, though proof (or truth) is built out of meaning.

The law of excluded middle is thus meaning-definite: the plays about it can be carried through. But with the intuitionist rule, *P* loses the play, so there is no winning strategy for *P*, i.e. there is no proof for it (there is also no strategy *against* it, as it would require *O* to win with the copy-cat rule); with the classical rule there is a winning strategy for *P* (reduced to the play, *O* has no choice to make) and so there is a proof for it. In intuitionistic logic, the law of excluded middle is meaning-definite but not truth-definite (if being true is having a proof for it, and being false having a proof that it entails a contradiction). It is however meaning-constitutive if meaning is dialogue-definiteness, that is, win or loss after finitely many moves in a game about a proposition. There is no other option besides win or loss, and if a player does not win, that player has lost.

### §5. Meaning and existence in the dialogical framework

We have up to now separated the level of meaning (the play level) from the level of proof and truth (the strategy level). But where does existence come into play? We could say that existence comes with a proof or with logical truth: once we have a demonstration that a proposition is absolutely true, then necessarily it is the case, and this factual aspect is existence, *being-there*. But this is an indirect justification for the level of proof and truth (strategies) to concern existence. I would like to provide a more direct link between the level of strategies and existence through the constructed fact that *there exists* a winning strategy for $P$ for a proposition.

In the dialogical framework, universal and existential quantifiers are similar to conjunction and disjunction: for the universal quantifier, it is the challenger who chooses the instance the other player has to implement in order to defend his universal statement, whereas for the existential quantifier, it is the defender who chooses the instance to implement.[74]

The existential quantifier is however not the appropriate notion of existence for separating the level of existence from that of meaning. The notion of existence looked for is rather the one stressed by Sundholm.

> "The meaning of the existential quantifier ∃ is accordingly explained in terms of the conditions under which $(\exists\, x \in D)\ P\,(x)$ is true.
> On the truth-maker analysis this truth condition is analysed as:
> there exists a truth-maker for $(\exists\, x \in D)\ P\,(x)$. [...] The notion of existence at issue here is not that of the existential quantifier."[75]

This notion of existence that is not grasped by the existential quantifier is the one we find when saying "there exists a winning strategy for $P$". The act of constructing the winning strategy out of all the relevant plays is what produces the "wholly new idea" of existence.

The level of strategies thus yields existence in a direct way: existence of the proof for the proposition. It is the existence of an activity, that of the winning strategy (which is more of the nature of an act than of an object: a winning strategy is knowing how to play in order to win whatever the opponent may play). This reinforces what has been previously said: it is the intuitionistic logic which is more adequate to the philosophical principles of the dialogical framework (intuitionists preferring to speak about activities rather than objects), which dovetails with Weyl's separation of processes as ongoing (dynamic take on infinity) and processes as results (static take on infinity), the first being the intuitionistic perspective, the second being rather the classical perspective.

But we especially recuperate Weyl's considerations on the nature of judgments. There are for him proper judgments such as "2 is an even number". There are universal statements which are instructions for judgments such as "all even numbers are dividable by 2", giving out the instruction that for a given number, such as 4, you may say that "4 is

74 Again, this is close to Weyl's notions of Freedom and Law.

75 G. Sundholm, "Existence, Proof and Truth-Making: A Perspective on the Intuitionistic Conception of Truth", *Topoi, 13*(2), 1994, pp. 118-119.

dividable by 2". Finally, there are existential statements which are abstracts either of proper judgments or judgment instructions. But we will not focus on the propositions the plays are *about* (and thus the notion of existence that is conveyed by the existential quantifier), but on the activity that the plays themselves *are*: the notion of existence here is the production of a process that has been carried through.

Indeed, winning strategies are indeed built out of *all* the relevant plays, so that "there exists" a winning strategy is constructed out of the instruction of taking each actual play. Each actual play, once it has been carried out, would be analogue to Weyl's proper judgment; the procedure for building a winning strategy is building an instruction out of the plays, which has meaning only when carried out. And from there we abstract, saying "there is" a winning strategy, which has meaning only insofar as we have the procedure to carry out the plays where individual actual choices are made (analogous to Weyl's proper judgments), in which *P* can win each time following the winning strategy. But the *there is* in itself is abstract. What Weyl say of sequences can be said of plays and winning strategies:

> "We are thus not at all talking of the *possibility* of construction. Such an existential claim is rather only made in view of the *achieved construction* of the *given proof*."[76]

## Conclusion

Against the problematic interrelation in intuitionistic logic of proof and meaning, I have tried to argue in favour of the separation of the level of meaning and the level of proof. The overall strategy I have adopted was 1. to examine Weyl's texts on their own in the first part, interpreting them as separating the level of meaning and the level of existence and defending this interpretation without importing considerations from the dialogical framework; and then 2. in the second part show how the dialogical framework provides the means to fully separate the level of meaning and the level of existence and proof. The dialogical framework is in this fashion a good tool for approaching philosophical texts, even if the proposed interpretation needs to be justified through the texts themselves and not through this logical framework, which is often exterior and posterior to the texts considered.

The dialogical framework allows for a clear separation between the level of meaning and the level of existence. Meaning is constituted by dialogue-definiteness, the fact that

76H. Weyl, "On the New Foundational Crisis of Mathematics" (1921), op. cit., p. 95.

plays about a proposition can be carried through in a finite number of moves with one player winning and the other losing at the end. Existence is reached at the level of strategies where something more is added to meaning (the plays) as the activity of building a winning strategy is carried through: saying *there exists a winning strategy* introduces existence in the form of an abstract that requires actual plays and a procedure on plays to carry out in order for it to be a meaningful abstract. Existence and meaning are distinct levels of consideration, and existential statements such as *there exists a winning strategy* require the level of meaning in order not to be pure meaningless abstracts.

But a winning strategy in the dialogical framework is equivalent to having a proof, and so between what has meaning and what can be proved (or refuted) there is the full range of contingent truths or yet unknown facts: these fall outside of the domain of what can be proved, but the dialogical framework provides the means for talking about them and showing they are meaningful independently of any proof.

The distinction between existence and meaning is operative in the dialogical framework; it is thus an external warrant for the proposed reading of Weyl's philosophy of mathematics which puts much stress on the distinction between a static conception of infinity (or a process in general) and a dynamic one. If this reading of Weyl is correct, the dynamic conception yields for Weyl the intuitionist way of doing mathematics and requires the distinction between existence and meaning. The law of excluded middle would then be operative in meaning-constitution (clear and unambiguous meaning), but it would not be truth-definite in an infinite context when the dynamic conception of infinity is adopted.